

A CHARACTER FORMULA FOR THE CATEGORY $\tilde{\mathcal{O}}$

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ABSTRACT. One may construct, for any function on the integers, an irreducible module of level zero for affine $\mathfrak{sl}(2)$, using the values of the function as structure constants. The modules constructed using exponential-polynomial functions realise the irreducible modules with finite-dimensional weight spaces in the category $\tilde{\mathcal{O}}$ of Chari. In this work, an expression for the formal character of such a module is derived using the highest-weight theory of truncations of the loop algebra.

1. INTRODUCTION

Let \mathfrak{g} denote the Lie algebra $\mathfrak{sl}(2)$ of 2×2 traceless matrices over an algebraically closed field \mathbb{k} of characteristic zero. Associated to \mathfrak{g} is the centreless affine Lie algebra $\tilde{\mathfrak{g}}$,

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{k}[t, t^{-1}] \oplus \mathbb{k}d$$

obtained as an extension of the loop algebra by a degree derivation d . The Cartan subalgebra of $\tilde{\mathfrak{g}}$ is given by $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{k}d$, where \mathfrak{h} is the Cartan subalgebra of \mathfrak{g} . The category $\tilde{\mathcal{O}}$, introduced by Chari [4], is an analogue of the category \mathcal{O} of Bernstein, Gelfand and Gelfand [2], corresponding to the natural Borel subalgebra of $\tilde{\mathfrak{g}}$ (that is, to the loop algebra of the Borel subalgebra of \mathfrak{g}). For every exponential-polynomial function $\varphi : \mathbb{Z} \rightarrow \mathbb{k}$, one may construct an irreducible $\tilde{\mathfrak{g}}$ -module $\mathbf{N}(\varphi)$ in the category $\tilde{\mathcal{O}}$ with finite-dimensional weight spaces. It follows from Chari [4], and from Billig and Zhao [3], that any irreducible module in the category $\tilde{\mathcal{O}}$ with all weight spaces of finite dimension can be obtained as the tensor product of a one-dimensional module with a module $\mathbf{N}(\varphi)$. In this paper, a formula for the character of the modules $\mathbf{N}(\varphi)$ is derived, thus describing the character of the irreducible modules in the category $\tilde{\mathcal{O}}$ with finite-dimensional weight spaces.

For any weight $\tilde{\mathfrak{g}}$ -module M , write

$$M = \bigoplus_{\chi \in \tilde{\mathfrak{h}}^*} M_\chi \quad \text{where} \quad h|_{M_\chi} = \chi(h), \quad h \in \tilde{\mathfrak{h}},$$

and if the weight spaces M_χ are finite dimensional, write

$$\text{char } M = \sum_{\chi \in \tilde{\mathfrak{h}}^*} \dim M_\chi Z^\chi$$

for its character.

A function $\varphi : \mathbb{Z} \rightarrow \mathbb{k}$ is *exponential polynomial* if it can be written as a finite sum of products of polynomial and exponential functions. For any $\lambda \in \mathbb{k}^\times$, define the exponential function

$$\text{EXP}(\lambda) : \mathbb{Z} \rightarrow \mathbb{k}, \quad \text{EXP}(\lambda)(m) = \lambda^m, \quad m \in \mathbb{Z}.$$

For any positive integer r , define the function

$$\wp_r = \sum_{\zeta^r=1} \text{EXP}(\zeta),$$

where the sum is over all roots of unity ζ such that $\zeta^r = 1$. If $\varphi : \mathbb{Z} \rightarrow \mathbb{k}$ is a non-zero exponential-polynomial function, then there exists a unique $r > 0$ such that

$$(1.1) \quad \varphi = \wp_r \cdot \sum_i a_i \text{EXP}(\lambda_i)$$

for some finite collection of polynomial functions $a_i : \mathbb{Z} \rightarrow \mathbb{k}$ and scalars $\lambda_i \in \mathbb{k}^\times$, such that if $(\lambda_i/\lambda_j)^r = 1$, then $i = j$. Let α denote the positive root of \mathfrak{g} , and let δ denote the fundamental imaginary root of $\tilde{\mathfrak{g}}$. The characters of the modules $\mathbf{N}(\varphi)$ are described by the following theorem.

Theorem. Let $\varphi : \mathbb{Z} \rightarrow \mathbb{k}$ be a non-zero exponential-polynomial function. In the notation of (1.1),

$$(1.2) \quad \text{char } \mathbf{N}(\varphi) = Z^{\varphi(0)\frac{\alpha}{2}} \cdot \frac{1}{r} \sum_{n \in \mathbb{Z}} \sum_{d|r} c_d(n) \left(P_\varphi(Z^{-d\alpha}) \right)^{\frac{r}{d}} Z^{n\delta},$$

where the inner sum is over the positive divisors d of r , the quantities $c_d(n)$ are Ramanujan sums, and

$$P_\varphi(Z) = \frac{\prod_{a_i \in \mathbb{Z}_+} (1 - Z^{a_i+1})}{(1 - Z)^M},$$

where $M = \sum_i (\deg a_i + 1)$ and the product is over those indices i for which $a_i \in \mathbb{Z}_+$.

The *Ramanujan sum* $c_d(n)$ is given by

$$(1.3) \quad c_d(n) = \frac{\phi(d)\mu(d')}{\phi(d')}, \quad d' = \frac{d}{\gcd(d, n)},$$

where ϕ denotes Euler's totient function and μ denotes the Möbius function.

It may be deduced from the character formula (1.2) that if the weight space $\mathbf{N}(\varphi)_\chi$ is non-zero, then

$$\chi = \varphi(0)\frac{\alpha}{2} - k\alpha + n\delta,$$

for some $n \in \mathbb{Z}$ and $k \geq 0$. The function $c_d(\cdot)$ has period d , and so for any $k \geq 0$, the multiplicity function

$$n \mapsto \dim \mathbf{N}(\varphi)_{\varphi(0)\frac{\alpha}{2} - k\alpha + n\delta}, \quad n \in \mathbb{Z},$$

has period r . Therefore the character of $\mathbf{N}(\varphi)$ is completely described by the array of weight-space multiplicities

$$\left[\dim \mathbf{N}(\varphi)_{\varphi(0)\frac{\alpha}{2} - k\alpha + n\delta} \right]$$

where $k \geq 0$ and $0 \leq n < r$. Examples of these arrays, such as those illustrated by Figures 1(a) – 1(d), may be computed in a straightforward manner using the formula (1.2). Columns are indexed left to right by n , where $0 \leq n < r$, while rows are indexed from top to bottom by $k \geq 0$.

Greenstein [9] (see also [6, Section 4.1]) has derived an explicit formula for the character of an integrable irreducible object of the category $\tilde{\mathcal{O}}$. These objects are precisely the exponential-polynomial modules $\mathbf{N}(\varphi)$ where φ is a linear combination of exponential functions with non-negative integral coefficients. Indeed, our result may alternatively be deduced by considering separately the case where $\mathbf{N}(\varphi)$ is integrable, employing the result of Greenstein, and the case where $\mathbf{N}(\varphi)$ is not integrable, using Molien's Theorem. Our approach, via a general study of finite cyclic-group actions, has the advantage of permitting a unified proof. Both approaches employ the explicit expression of the character of an irreducible highest-weight module for a truncated current Lie algebra described in [13].

1	0	0	0	0	0	1	0	0	0
1	1	1	1	1	1	2	2	2	2
3	2	3	2	3	2	10	8	10	8
4	3	3	4	3	3	30	30	30	30
3	2	3	2	3	2	86	80	84	80
1	1	1	1	1	1	198	198	198	198
1	0	0	0	0	0	434	424	434	424
0	0	0	0	0	0	858	858	858	858
\vdots					\vdots	\vdots			\vdots
(a) $\varphi = \wp_6$						(b) $\varphi = -\wp_4(\text{EXP}(\lambda) + \text{EXP}(\mu))$			
1	0	0	0	0	0	1	0		
1	1	1	1	1	1	2	2		
4	3	4	3	4	3	5	3		
10	9	9	10	9	9	6	6		
22	20	22	20	22	20	9	7		
42	42	42	42	42	42	10	10		
80	75	78	76	78	75	13	11		
132	132	132	132	132	132	14	14		
217	212	217	212	217	212	17	15		
335	333	333	335	333	333	18	18		
\vdots					\vdots	\vdots	\vdots		
(c) $\varphi = -\wp_6$						(d) $\varphi = \wp_2(\text{EXP}(\lambda) - \text{EXP}(\mu))$			

Figure 1: Array of weight-space multiplicities of $\mathbf{N}(\varphi)$

2. EXPONENTIAL-POLYNOMIAL FUNCTIONS

A function $\varphi : \mathbb{Z} \rightarrow \mathbb{k}$ is *exponential polynomial* if it can be written as a finite sum of products of polynomial and exponential functions, i.e.

$$\sum_{\lambda \in \mathbb{k}^\times} \varphi_\lambda \text{EXP}(\lambda),$$

for some polynomial functions $\varphi_\lambda : \mathbb{Z} \rightarrow \mathbb{k}$ and distinct scalars $\lambda \in \mathbb{k}^\times$. Write

$$\mathcal{E} = \{ \varphi : \mathbb{Z} \rightarrow \mathbb{k} \mid \varphi \text{ is exponential polynomial} \}.$$

Let $\mathcal{A} = \mathbb{k}[t, t^{-1}]$.

2.1. Module structure. Let $\mathcal{F} = \{ \varphi : \mathbb{Z} \rightarrow \mathbb{k} \}$. Define an endomorphism τ of the vector space \mathcal{F} via

$$(\tau \cdot \varphi)(m) = \varphi(m+1), \quad m \in \mathbb{Z}, \quad \varphi \in \mathcal{F}.$$

The rule $t \mapsto \tau$ endows \mathcal{F} with the structure of an \mathcal{A} -module. For any $k \geq 0$ and $\lambda \in \mathbb{k}^\times$, define the function $\theta_{\lambda,k} \in \mathcal{F}$ by

$$\theta_{\lambda,k}(m) = m^k \lambda^m, \quad m \in \mathbb{Z}.$$

By definition, \mathcal{E} is the linear subspace of \mathcal{F} spanned by the $\theta_{\lambda,k}$. It is immediate from the following lemma that \mathcal{E} is an \mathcal{A} -submodule of \mathcal{F} .

Lemma 2.1. For any $\lambda, \mu \in \mathbb{k}^\times$ and $k \geq 0$,

- i. $(t - \mu) \cdot \theta_{\lambda,k} = (\lambda - \mu)\theta_{\lambda,k} + \lambda \sum_{j=0}^{k-1} \binom{k}{j} \theta_{\lambda,j}$;
- ii. $(t - \lambda)^k \cdot \theta_{\lambda,k} = k! \lambda^k \theta_{\lambda,0}$.
- iii. $(t - \lambda)^{k+1} \cdot \theta_{\lambda,k} = 0$.

Proof. For any $m \in \mathbb{Z}$,

$$\begin{aligned} (t \cdot \theta_{\lambda,k})(m) &= \theta_{\lambda,k}(m+1) = (m+1)^k \lambda^{m+1} \\ &= \lambda \sum_{j=0}^k \binom{k}{j} m^j \lambda^m \\ &= \lambda \sum_{j=0}^k \binom{k}{j} \theta_{\lambda,j}(m). \end{aligned}$$

Therefore,

$$(t - \mu) \cdot \theta_{\lambda,k} = \lambda \sum_{j=0}^k \binom{k}{j} \theta_{\lambda,j} - \mu \theta_{\lambda,k},$$

and so part (i) is proven. Part (ii) is proven by induction. The claim is trivial if $k = 0$, so suppose that the claim holds for some $k \geq 0$. Then

$$\begin{aligned} (t - \lambda)^{k+1} \cdot \theta_{\lambda,k+1} &= (t - \lambda)^k \cdot \lambda \sum_{j=0}^k \binom{k+1}{j} \theta_{\lambda,j} \\ &= \lambda \binom{k+1}{k} k! \lambda^k \theta_{\lambda,0} \quad (\text{by inductive hypothesis}) \\ &= (k+1)! \lambda^{k+1} \theta_{\lambda,0}, \end{aligned}$$

where part (i) is used in obtaining the first and second equalities. Therefore the claim holds for all $k \geq 0$ by induction. Part (iii) follows immediately from parts (i) and (ii). \square

Proposition 2.2. The set $\{\theta_{\lambda,k} \mid \lambda \in \mathbb{k}^\times, k \geq 0\}$ is linearly independent, and hence is a basis for \mathcal{E} .

Proof. Suppose that $\gamma_{\lambda,k} \in \mathbb{k}$, $\lambda \in \mathbb{k}^\times$, $k \geq 0$, are scalars such that the sum

$$\varphi = \sum_{\lambda \in \mathbb{k}^\times} \sum_{k \geq 0} \gamma_{\lambda,k} \theta_{\lambda,k},$$

is finite and equal to zero. Write $Z = \{\lambda \in \mathbb{k}^\times \mid \gamma_{\lambda,k} \neq 0 \text{ for some } k \geq 0\}$, and let

$$n_\lambda = \max\{k \mid \gamma_{\lambda,k} \neq 0\}, \quad \lambda \in Z.$$

Then, for any $\lambda \in Z$,

$$\begin{aligned} 0 &= \prod_{\mu \in Z} (t - \mu)^{n_\mu + 1 - \delta_{\lambda,\mu}} \cdot \varphi \\ &= \prod_{\mu \in Z} (t - \mu)^{n_\mu + 1 - \delta_{\lambda,\mu}} \cdot (\gamma_{\lambda,n_\lambda} \theta_{\lambda,n_\lambda}) \\ &= \gamma_{\lambda,n_\lambda} \lambda^{n_\lambda} n_\lambda! \prod_{\mu \in Z, \mu \neq \lambda} (\lambda - \mu)^{n_\mu + 1} \cdot \theta_{\lambda,0}, \end{aligned}$$

by Lemma 2.1. Therefore $\gamma_{\lambda,n_\lambda} = 0$ for all $\lambda \in Z$, which is absurd, unless Z is the empty set. \square

2.2. Recurrence relations. Suppose that $c(t) \in \mathbb{k}[t]$ is a non-zero polynomial of degree q , and write $c(t) = \sum_{k=0}^q c_k t^k$. Then, for any $\varphi \in \mathcal{F}$, we have $c \cdot \varphi = 0$ if and only if

$$(2.3) \quad 0 = (c \cdot \varphi)(m) = c_0 \varphi(m) + c_1 \varphi(m+1) + \cdots + c_q \varphi(m+q),$$

for all $m \in \mathbb{Z}$. That is, $c \cdot \varphi = 0$ precisely when the values of φ satisfy the linear homogeneous recurrence relation with constant coefficients (2.3) defined by c . The following proposition characterises the exponential-polynomial functions as the solutions of such recurrence relations.

Proposition 2.4. The \mathcal{A} -submodule $\mathcal{E} \subset \mathcal{F}$ is characterised by

$$\mathcal{E} = \{\varphi \in \mathcal{F} \mid c \cdot \varphi = 0 \text{ for some } c \in \mathbb{k}[t]\}.$$

Proof. Suppose that $c \in \mathbb{k}[t]$ is of degree q . The equation $c \cdot \varphi = 0$ is equivalent to the relation (2.3), and so the space consisting of all solutions φ is at most q -dimensional. Now write $Z \subset \mathbb{k}^\times$ for the set of all roots of c . The field \mathbb{k} is algebraically closed, and so

$$(2.5) \quad c(t) \sim_{\mathbb{k}^\times} \prod_{\lambda \in Z} (t - \lambda)^{m_\lambda},$$

where m_λ is the multiplicity of the root $\lambda \in Z$. Lemma 2.1 shows that the set

$$\{ \theta_{\lambda,k} \mid \lambda \in Z, \ 0 \leq k < m_\lambda \},$$

which is of size $\sum_{\lambda \in Z} m_\lambda = q$, consists of solutions to $c \cdot \varphi = 0$. By Proposition 2.2, this set is linearly independent, and hence is a basis for the solution space. This proves the inclusion \supset . The inclusion \subset follows immediately from Lemma 2.1 part (iii). \square

2.3. Characteristic polynomials. By Proposition 2.4, for any $\varphi \in \mathcal{E}$, the annihilator $\text{ann}(\varphi) \subset \mathbb{k}[t]$ is a non-zero ideal of $\mathbb{k}[t]$. The unique monic generator $c_\varphi \in \text{ann}(\varphi)$ is called the *characteristic polynomial* of φ .

Proposition 2.6. Suppose that $\varphi \in \mathcal{E}$ and write

$$(2.7) \quad \varphi = \sum_{\lambda \in \mathbb{k}^\times} \varphi_\lambda \text{EXP}(\lambda),$$

as a finite sum of products of polynomials functions φ_λ and exponential functions $\text{EXP}(\lambda)$, $\lambda \in \mathbb{k}^\times$. Then

$$(2.8) \quad c_\varphi(t) = \prod_{\lambda \in Z} (t - \lambda)^{\deg \varphi_\lambda + 1},$$

where $Z = \{ \lambda \in \mathbb{k}^\times \mid \varphi_\lambda \neq 0 \}$.

Proof. Suppose that $\varphi \in \mathcal{E}$ has the form (2.7), let $c \in \mathbb{k}[t]$ be non-zero, and write c in the form (2.5). By Lemma 2.1, we have $c \cdot \varphi = 0$ if and only if $m_\lambda > \deg \varphi_\lambda$ whenever $\varphi_\lambda \neq 0$. The polynomial (2.8) is the minimal degree monic polynomial that satisfies this condition, and hence is the characteristic polynomial. \square

2.4. Degree. It follows from the characterisation of \mathcal{E} given in subsection 2.2 that if $\varphi \in \mathcal{E}$ is non-zero, then the support of φ is not wholly contained in either of the infinite subsets of consecutive integers $\mathbb{N}, -\mathbb{N} \subset \mathbb{Z}$. Lemma 2.9 demonstrates that the monoid generated by the support of φ is of the form $r\mathbb{Z}$, for some unique positive integer r . Write $\deg \varphi = r$ for the *degree* of φ .

Lemma 2.9. Suppose that A is a submonoid of \mathbb{Z} such that $\mathbb{N}, -\mathbb{N} \not\subset A$. Then $A = r\mathbb{Z}$, where $r \in A$ is any non-zero element of minimal absolute value.

Proof. Let $r \in A \cap \mathbb{N}$ be of minimal absolute value. For any $m \in A \cap -\mathbb{N}$, we have that $m + kr \in A$ where k is the unique positive integer such that

$$0 \leq m + kr < r.$$

Thus $m + kr = 0$ by the minimality of r ; it follows that r divides m , for any $m \in A \cap -\mathbb{N}$. Moreover,

$$-r = m + (k - 1)r \in A$$

since $k - 1$ is non-negative. It follows therefore that $-r$ is the element of minimal absolute value in $A \cap -\mathbb{N}$. The argument above with inequalities reversed shows that $-r$ divides all positive elements of A , and so $A \subset r\mathbb{Z}$. The opposite inclusion is obvious since $r, -r \in A$ and A is closed under addition. \square

2.5. Expression. For any positive integer r , denote by \mathbb{Z}_r the additive group of integers considered modulo r , by $\mathfrak{R}(r)$ the set of primitive roots of unity of order r , and by ζ_r some fixed element of $\mathfrak{R}(r)$.

Lemma 2.10. Suppose that $\varphi \in \mathcal{E}$ is non-zero and that $\deg \varphi = r$. Then $r > 0$, and $\varphi_\lambda = \varphi_{\zeta\lambda}$ whenever $\lambda, \zeta \in \mathbb{k}^\times$ and $\zeta^r = 1$. Moreover, there exists $\psi \in \mathcal{E}$ such that

- i. $\varphi = \wp_r \psi$, and
- ii. $c_\varphi = \prod_{i \in \mathbb{Z}_r} c_\psi(\zeta_r^i t)$ is a decomposition of c_φ into co-prime factors.

Proof. The support of φ is contained in $r\mathbb{Z}$. If $r = 0$, then $\varphi(m) = 0$ for any non-zero $m \in \mathbb{Z}$. In particular φ has infinitely many consecutive zeros, and so $\varphi = 0$, contrary to hypothesis. Therefore $r > 0$. The support of φ is contained in the support of \wp_r , and so

$\varphi = \frac{1}{r} \wp_r \varphi$. Hence

$$\varphi_\lambda = \left(\frac{1}{r} \wp_r \varphi\right)_\lambda = \frac{1}{r} \sum_{i \in \mathbb{Z}_r} \varphi(\zeta_r^i \lambda),$$

for any $\lambda \in \mathbb{k}^\times$. If $\zeta^r = 1$, then the expression on the right-hand side is invariant under the substitution $\lambda \mapsto \zeta \lambda$, and so the first claim is proven.

Multiplication by ζ_r decomposes \mathbb{k}^\times into a disjoint union of orbits for the cyclic group \mathbb{Z}_r , and all orbits are of size r . Choose any set B of representatives, so that $\mathbb{k}^\times = \bigsqcup_{i \in \mathbb{Z}_r} \zeta_r^i B$. Then $\psi = \sum_{\lambda \in B} \varphi_\lambda \text{EXP}(\lambda)$ has the required property, by Proposition 2.6. \square

Remark 2.11. The function $\psi \in \mathcal{E}$ of Lemma 2.10 is not unique. Indeed, if

$$\psi = \sum_i a_i \text{EXP}(\mu_i)$$

has the required property, then so does $\psi' = \sum_i a_i \text{EXP}(\zeta_r^{n_i} \mu_i)$ for any $n_i \in \mathbb{Z}_r$.

3. THE CATEGORY $\tilde{\mathcal{O}}$

For any Lie algebra \mathfrak{a} over \mathbb{k} , denote by

$$\hat{\mathfrak{a}} = \mathfrak{a} \otimes \mathbb{k}[t, t^{-1}]$$

the *loop algebra* associated to \mathfrak{a} , with the Lie bracket

$$(3.1) \quad [x \otimes t^i, y \otimes t^j] = [x, y] \otimes t^{i+j}, \quad x, y \in \mathfrak{a}, \quad i, j \in \mathbb{Z}.$$

Let $\{e, f, h\}$ be a standard basis for $\mathfrak{g} = \mathfrak{sl}(2)$, so that

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

The Lie algebra \mathfrak{g} has a triangular decomposition

$$(3.2) \quad \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-, \quad e \in \mathfrak{g}_+, \quad h \in \mathfrak{h}, \quad f \in \mathfrak{g}_-.$$

The decomposition (3.2) defines a decomposition of the loop algebra $\hat{\mathfrak{g}}$

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{g}}_-$$

as a direct sum of subalgebras. The centreless affine Lie algebra $\tilde{\mathfrak{g}}$ is the one-dimensional extension of $\hat{\mathfrak{g}}$ by $\mathbb{k}d$ defined by

$$[d, x \otimes t^n] = nx \otimes t^n, \quad x \in \mathfrak{g}, \quad n \in \mathbb{Z}.$$

Let $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{k}d$, and let $\alpha, \delta \in \tilde{\mathfrak{h}}^*$ be given by

$$\alpha(h) = 2, \quad \alpha(d) = 0, \quad \delta(h) = 0, \quad \delta(d) = 1.$$

Denote by \mathfrak{t} the Heisenberg subalgebra $\hat{\mathfrak{h}} \oplus \mathbb{k}d$ of $\tilde{\mathfrak{g}}$.

For any weight $\tilde{\mathfrak{g}}$ -module M , write

$$s(M) = \{ \chi \in \tilde{\mathfrak{h}}^* \mid M_\chi \neq 0 \}.$$

The category $\tilde{\mathcal{O}}$, introduced by Chari [4], consists of the weight $\tilde{\mathfrak{g}}$ -modules M such that

$$s(M) \subset \bigcup_{\lambda \in A} (\lambda - \mathbb{Z}_+ \alpha \times \mathbb{Z} \delta)$$

for some finite subset $A = A_M \subset \mathfrak{h}^*$. The morphisms of the category are the homomorphisms of $\tilde{\mathfrak{g}}$ -modules.

3.1. The modules $\mathbf{N}(\varphi)$. Let $\varphi \in \mathcal{E}$. It is straightforward to verify that \mathcal{A} is a \mathfrak{t} -module via

$$h \otimes t^m \cdot t^n = \varphi(m) t^{m+n}, \quad d \cdot t^n = n t^n, \quad m, n \in \mathbb{Z}.$$

Denote by $\mathbf{H}(\varphi)$ the \mathfrak{t} -submodule generated by $1 \in \mathcal{A}$.

Lemma 3.3. For any non-zero $\varphi \in \mathcal{E}$,

- i. $\mathbf{H}(\varphi) = \mathbb{k}[t^r, t^{-r}]$, where $r = \deg \varphi$;
- ii. $\mathbf{H}(\varphi)$ is irreducible.

Proof. By definition, $\mathbf{H}(\varphi)$ is spanned by monomials

$$(3.4) \quad h \otimes t^{m_1} \cdots h \otimes t^{m_k} \cdot 1 = \prod_{i=1}^k \varphi(m_i) t^{m_1 + \cdots + m_k},$$

where $k \geq 0$ and $m_i \in \mathbb{Z}$. Therefore $t^m \in \mathbf{H}(\varphi)$ precisely when $m = m_1 + \cdots + m_k$ for some m_i in the support of φ . That is, $t^m \in \mathbf{H}(\varphi)$ if and only if $m \in A$ where A is the monoid generated by the support of φ . By Lemma 2.9, $A = r\mathbb{Z}$, where $r = \deg \varphi$, and so $\mathbf{H}(\varphi) = \mathbb{k}[t^r, t^{-r}]$, proving part (i).

Now suppose that $v \in \mathbf{H}(\varphi)$ is a non-zero weight vector. Then v is proportional to t^m for some $m \in \mathbb{Z}$. By part (i), $t^{-m} \in \mathbf{H}(\varphi)$ also, and so there exist integers m_1, \dots, m_k in the

support of φ such that t^{-m} is proportional to a monomial (3.4). Then

$$h \otimes t^{m_1} \cdots h \otimes t^{m_k} \cdot t^m = \prod_{i=1}^k \varphi(m_i) t^0$$

is proportional to the generator $1 \in \mathbf{H}(\varphi)$. Thus $\mathbf{H}(\varphi)$ is generated by any non-zero weight vector v , and hence is irreducible. \square

For any $\varphi \in \mathcal{E}$, let $\hat{\mathfrak{g}}_+ \cdot \mathbf{H}(\varphi) = 0$. Denote by $\mathbf{N}(\varphi)$ the unique irreducible quotient of the induced module

$$\text{Ind}_{t+\hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} \mathbf{H}(\varphi).$$

3.2. Irreducible modules in $\tilde{\mathcal{O}}$. For any $\gamma \in \mathbb{k}$, denote by $\mathbb{k}u_\gamma$ the one-dimensional $\tilde{\mathfrak{g}}$ -module given by

$$\hat{\mathfrak{g}} \cdot u_\gamma = 0, \quad d \cdot u_\gamma = \gamma u_\gamma.$$

It is apparent from subsection 3.1 that one may in fact construct an irreducible $\tilde{\mathfrak{g}}$ -module $\mathbf{N}(\varphi)$ for any $\varphi \in \mathcal{F}$ such that the monoid generated by the support of φ has the form $r\mathbb{Z}$, for some non-negative integer r . Chari [4] demonstrates that any irreducible object of the category $\tilde{\mathcal{O}}$ can be obtained as the tensor product of such a module $\mathbf{N}(\varphi)$ with a one-dimensional module $\mathbb{k}u_\gamma$. On the other hand, it follows from the work of Billig and Zhao [3] that $\mathbf{N}(\varphi)$ has all weight spaces finite dimensional precisely when $\varphi \in \mathcal{E}$. Thus:

Theorem 3.5. [3, 4] Suppose that M is an irreducible module in the category $\tilde{\mathcal{O}}$, and that all weight spaces of M are finite dimensional. Then $M \cong \mathbf{N}(\varphi) \otimes \mathbb{k}u_\gamma$ for some $\varphi \in \mathcal{E}$ and $\gamma \in \mathbb{k}$.

4. LOOP-MODULE REALISATION

For any $\hat{\mathfrak{g}}$ -module M , denote by

$$\widehat{M} = M \otimes \mathbb{k}[t, t^{-1}]$$

the $\tilde{\mathfrak{g}}$ -module defined by

$$x \otimes t^m \cdot v \otimes t^n = (x \otimes t^m \cdot v) \otimes t^{m+n}, \quad d \cdot v \otimes t^n = nv \otimes t^n,$$

for any $m, n \in \mathbb{Z}$, $v \in M$, and $x \in \mathfrak{g}$. Modules thus constructed are called *loop modules*. For $\varphi \in \mathcal{E}$, let $\mathbb{k}v_\varphi$ be the one-dimensional $\hat{\mathfrak{h}}$ -module defined by

$$\mathfrak{h} \otimes a \cdot v_\varphi = (a \cdot \varphi)(0)v_\varphi, \quad a \in \mathcal{A}.$$

Let $\hat{\mathfrak{g}}_+ \cdot v_\varphi = 0$, let

$$\mathfrak{V}(\varphi) = \text{Ind}_{\hat{\mathfrak{h}} + \hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} \mathbb{k}v_\varphi$$

denote the induced $\hat{\mathfrak{g}}$ -module, and let $\mathfrak{L}(\varphi)$ denote the unique irreducible quotient of $\mathfrak{V}(\varphi)$. In this section, it is shown that if $\varphi \in \mathcal{E}$ is non-zero and $r = \deg \varphi$, then $\mathbf{N}(\varphi)$ is isomorphic to a direct summand of the loop module $\widehat{\mathfrak{L}(\varphi)}$, and moreover that this summand may be described in terms of the semi-invariants of an action of the cyclic group \mathbb{Z}_r on $\mathfrak{L}(\varphi)$. The results of this section are due to Chari and Pressley [7] (see also [5]).

Denote by $\text{ord } \eta$ the order of a finite-order automorphism η . If η is an endomorphism of a vector space V , write

$$V|_\lambda^\eta = \{v \in V \mid \eta(v) = \lambda v\}$$

for the eigenspace of eigenvalue λ , for any $\lambda \in \mathbb{k}$.

4.1. Cyclic group action on $\mathfrak{L}(\varphi)$.

Lemma 4.1. Suppose that $\varphi \in \mathcal{E}$ is non-zero, and that $\zeta \in \mathbb{k}^\times$ is such that $\zeta^r = 1$, where $r = \deg \varphi$. Then for all $a \in \mathcal{A}$,

$$(a(\zeta t) \cdot \varphi)(0) = (a \cdot \varphi)(0).$$

Proof. The support of φ is contained in $r\mathbb{Z}$. Therefore, if $a(t) = \sum_i a_i t^i$, then

$$(a(\zeta t) \cdot \varphi)(0) = \sum_{i \equiv 0 \pmod{r}} a_i \zeta^i \varphi(i) = \sum_{i \equiv 0 \pmod{r}} a_i \varphi(i) = (a \cdot \varphi)(0). \quad \square$$

Proposition 4.2. Suppose that $\varphi \in \mathcal{E}$ is non-zero, and let $r = \deg \varphi$. Then there exists an order- r automorphism $\eta = \eta_\varphi$ of the \mathfrak{h} -module $\mathfrak{L}(\varphi)$ defined by $\eta(v_\varphi) = v_\varphi$ and

$$\eta(x \otimes a \cdot w) = x \otimes a(\zeta^{-1}t) \cdot \eta(w), \quad x \in \mathfrak{g}, \quad a \in \mathcal{A}, \quad w \in \mathfrak{L}(\varphi),$$

where $\zeta = \zeta_r$. Moreover, η decomposes $\mathfrak{L}(\varphi)$ as a direct sum of eigenspaces

$$\mathfrak{L}(\varphi) = \bigoplus_{i \in \mathbb{Z}_r} \mathfrak{L}(\varphi)|_{\zeta^i}^\eta.$$

Proof. The rule $t \mapsto \zeta^{-1}t$ extends to an automorphism of \mathcal{A} , which defines an automorphism of the loop algebra $\hat{\mathfrak{g}}$. This automorphism in turn defines an automorphism η of the universal enveloping algebra $\mathcal{U}(\hat{\mathfrak{g}})$. The universal module $\mathfrak{V}(\varphi)$ may be realised as the quotient of $\mathcal{U}(\hat{\mathfrak{g}})$ by the left ideal I generated by $\hat{\mathfrak{g}}_+$ and by the elements of the set

$$\{h \otimes a - (a \cdot \varphi)(0) \mid a \in \mathcal{A}\}.$$

The map η preserves this set by Lemma 4.1:

$$\begin{aligned} \eta(h \otimes a - (a \cdot \varphi)(0)) &= h \otimes a(\zeta^{-1}t) - (a \cdot \varphi)(0) \\ &= h \otimes a(\zeta^{-1}t) - (a(\zeta^{-1}t) \cdot \varphi)(0). \end{aligned}$$

Clearly η preserves $\hat{\mathfrak{g}}_+$, and so $\eta(I) = I$. Therefore η is well-defined on the quotient $\mathfrak{V}(\varphi)$ of $\mathcal{U}(\hat{\mathfrak{g}})$. The monomial

$$f \otimes t^{n_1} \cdots f \otimes t^{n_k} \cdot v_\varphi \in \mathfrak{V}(\varphi)$$

is an eigenvector of eigenvalue ζ^{-m} where $m = \sum_{i=1}^k n_i$, and so the Poincaré-Birkhoff-Witt Theorem guarantees a decomposition

$$(4.3) \quad \mathfrak{V}(\varphi) = \bigoplus_{i \in \mathbb{Z}_r} \mathfrak{V}(\varphi)|_{\zeta^i}^\eta$$

of $\mathfrak{V}(\varphi)$ into eigenspaces for η . It is easy to check that η commutes with the action of $h \otimes t^0$, from which it follows that if U is a proper submodule, then so is $\eta(U)$. Hence η preserves the maximal submodule of $\mathfrak{V}(\varphi)$, and so is defined on the quotient $\mathfrak{L}(\varphi)$. This induced map is of order r , by construction, and decomposes $\mathfrak{L}(\varphi)$ in the manner claimed by (4.3). \square

4.2. Decomposition of a loop module. For any non-zero $\varphi \in \mathcal{E}$, define an automorphism $\hat{\eta}_\varphi$ of the vector space $\widehat{\mathfrak{L}(\varphi)}$ via

$$\hat{\eta}_\varphi(u \otimes a) = \eta_\varphi(u) \otimes a(\zeta_r t), \quad u \in \mathfrak{L}(\varphi), \quad a \in \mathcal{A},$$

where $r = \deg \varphi$.

Theorem 4.4. Suppose that $\varphi \in \mathcal{E}$ is non-zero. Let $r = \deg \varphi$, $\zeta = \zeta_r$ and $\hat{\eta} = \hat{\eta}_\varphi$. Then:

- i. $\hat{\eta}$ is automorphism of the $\tilde{\mathfrak{g}}$ -module $\widehat{\mathfrak{L}(\varphi)}$ of order r ;

ii. $\hat{\eta}$ decomposes $\widehat{\mathfrak{L}(\varphi)}$ as a direct sum of eigenspaces

$$\widehat{\mathfrak{L}(\varphi)} = \bigoplus_{i \in \mathbb{Z}_r} \widehat{\mathfrak{L}(\varphi)}|_{\zeta^i}^{\hat{\eta}},$$

where

$$\widehat{\mathfrak{L}(\varphi)}|_{\zeta^i}^{\hat{\eta}} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{L}(\varphi)|_{\zeta^{i-m}}^{\eta_\varphi} \otimes \mathfrak{t}^m, \quad i \in \mathbb{Z}_r;$$

iii. For any $i \in \mathbb{Z}_r$, the $\tilde{\mathfrak{g}}$ -modules $\widehat{\mathfrak{L}(\varphi)}|_{\zeta^i}^{\hat{\eta}}$ and $\mathbf{N}(\varphi) \otimes \mathbb{k}u_i$ are isomorphic.

Proof. For any $x \in \mathfrak{g}$, $u \in \mathfrak{L}(\varphi)$, $a \in \mathcal{A}$ and $m \in \mathbb{Z}$,

$$\begin{aligned} \hat{\eta}(x \otimes \mathfrak{t}^m \cdot u \otimes a) &= \hat{\eta}((x \otimes \mathfrak{t}^m \cdot u) \otimes \mathfrak{t}^m a) \\ &= \zeta^m \eta(x \otimes \mathfrak{t}^m \cdot u) \otimes \mathfrak{t}^m a(\zeta \mathfrak{t}) \\ &= \zeta^m \zeta^{-m} (x \otimes \mathfrak{t}^m \cdot \eta(u)) \otimes \mathfrak{t}^m a(\zeta \mathfrak{t}) \\ &= x \otimes \mathfrak{t}^m \cdot (\eta(u) \otimes a(\zeta \mathfrak{t})) \\ &= x \otimes \mathfrak{t}^m \cdot \hat{\eta}(u \otimes a), \end{aligned}$$

where $\eta = \eta_\varphi$. The map $\hat{\eta}$ is of order r by definition, and so part (i) is proven. Part (ii) follows immediately from Proposition 4.2. Let $i \in \mathbb{Z}_r$, and write

$$U = \widehat{\mathfrak{L}(\varphi)}|_{\zeta^i}^{\hat{\eta}}, \quad U' = \bigoplus_{m \in \mathbb{Z}} U_{\frac{\varphi(0)}{2}\alpha + m\delta}.$$

Then U' and $\mathbf{H}(\varphi) \otimes \mathbb{k}u_i$ are isomorphic as \mathfrak{t} -modules, via

$$v_\varphi \otimes \mathfrak{t}^{mr+i} \mapsto \mathfrak{t}^{mr} \otimes u_i, \quad m \in \mathbb{Z}.$$

This map extends uniquely to an epimorphism of $\tilde{\mathfrak{g}}$ -modules $U \rightarrow \mathbf{N}(\varphi) \otimes \mathbb{k}u_i$. Therefore it is sufficient to prove that U is an irreducible $\tilde{\mathfrak{g}}$ -module. Suppose that W is a submodule of U . Then W contains a non-zero maximal weight vector $v \otimes \mathfrak{t}^n$. The $\hat{\mathfrak{g}}$ -module epimorphism $U \rightarrow \mathfrak{L}(\varphi)$ that is induced by $\mathfrak{t} \mapsto 1$ maps this element to a non-zero maximal vector of $\mathfrak{L}(\varphi)$. Therefore $v = \lambda v_\varphi$ is a non-zero scalar multiple of the highest-weight vector. Hence W has non-trivial intersection with the generating subspace U' of U . The \mathfrak{t} -module U' is irreducible, so $U' \subset W$, and thus $W = U$. Therefore U is irreducible. \square

4.3. Characters and semi-invariants. If $\varphi \in \mathcal{E}$ is non-zero and $r = \deg \varphi$, then Theorem 4.4 describes the modules $\mathbf{N}(\varphi)$ in terms of the semi-invariants of $\mathfrak{L}(\varphi)$ with respect to the action of the cyclic group \mathbb{Z}_r defined by η . In particular, we have the following description of the character of $\mathbf{N}(\varphi)$.

Corollary 4.5. Suppose that $\varphi \in \mathcal{E}$ is non-zero and let $\deg \varphi = r$. Then

$$\text{char } \mathbf{N}(\varphi) = Z^{\frac{\varphi(0)}{2}\alpha} \cdot \sum_{k \geq 0} \sum_{n \in \mathbb{Z}} \dim \mathfrak{L}(\varphi)_{\frac{\varphi(0)}{2}\alpha - k\alpha} \Big|_{\zeta^n}^{\eta} Z^{-k\alpha + n\delta},$$

where $\zeta = \zeta_r$.

5. SEMI-INVARIANTS OF ACTIONS OF FINITE CYCLIC GROUPS

A \mathbb{Z}_+ -graded vector-space is a vector space V over \mathbb{k} with a decomposition $V = \bigoplus_{k \geq 0} V(k)$ of V into finite-dimensional subspaces indexed by \mathbb{Z}_+ . If V is a \mathbb{Z}_+ -graded vector space and r is a positive integer, then the tensor power

$$V^r := V \otimes \cdots \otimes V \quad (r \text{ times})$$

is also a \mathbb{Z}_+ -graded vector space, with the decomposition

$$V^r = \bigoplus_{k \geq 0} V^r(k), \quad V^r(k) = \bigoplus_{k_1 + \cdots + k_r = k} V(k_1) \otimes \cdots \otimes V(k_r).$$

The finite cyclic group \mathbb{Z}_r acts on V^r by cycling homogeneous tensors; the generator $1 \in \mathbb{Z}_r$ acts via the vector space automorphism

$$\sigma_r : v_1 \otimes \cdots \otimes v_r \mapsto v_r \otimes v_1 \otimes \cdots \otimes v_{r-1}, \quad v_i \in V,$$

and this action preserves the grading, so that $\sigma_r(V^r(k)) = V^r(k)$, for any $k \geq 0$. For any $U \subset V^r$, let

$$U_n = U|_{\zeta^n}^{\sigma_r}, \quad n \in \mathbb{Z}.$$

The automorphism σ_r decomposes V^r as a direct sum of \mathbb{Z}_+ -graded vector spaces

$$V^r = \bigoplus_{n \in \mathbb{Z}_r} V_n^r, \quad V_n^r = \bigoplus_{k \geq 0} V_n^r(k), \quad V_n^r(k) = (V^r(k))_n.$$

Associated to any \mathbb{Z}_+ -graded vector space U is the Poincaré series

$$\mathcal{P}_U(X) = \sum_{k \geq 0} \dim U_k X^k \in \mathbb{Z}_+[[X]].$$

Theorem. For any \mathbb{Z}_+ -graded vector space V , $r > 0$ and $n \in \mathbb{Z}$,

$$\mathcal{P}_{V_n^r}(X) = \frac{1}{r} \sum_{d|r} c_d(n) \left(\mathcal{P}_V(X^d) \right)^{\frac{r}{d}}.$$

In this section, we describe an elementary proof of this statement. If U is the regular representation of \mathbb{Z}_r and $V = S(U)$ is the symmetric algebra, then the statement follows easily from Molien's Theorem and the identity (5.2).

Let μ denote the Möbius function, i.e. the function

$$\mu : \mathbb{N} \rightarrow \{-1, 0, 1\}$$

such that $\mu(d) = (-1)^l$ if d is the product of l distinct primes, $l \geq 0$, and $\mu(d) = 0$ otherwise. For any $r > 0$, the function μ satisfies the fundamental property

$$(5.1) \quad \sum_{d|r} \mu(d) = \delta_{r,1},$$

where δ denotes the Kronecker function. A summation $\sum_{d|r} a_d$ is to be understood as the sum of all the a_d where d is a positive divisor of r . Let $\phi : \mathbb{N} \rightarrow \mathbb{N}$ denote Euler's totient function, so that

$$\phi(d) = \# \{ 0 < k \leq d \mid \gcd(k, d) = 1 \}, \quad d > 0.$$

For any positive integer d and $n \in \mathbb{Z}$, the quantity $c_d(n)$ defined by (1.3) is called a *Ramanujan sum*, a *von Sterneck function*, or a *modified Euler number*. These quantities have extensive applications in number theory (see, for example, [12], [10]), although we require only the most basic properties, such as those described in [8]. In particular, we note the identities

$$(5.2) \quad c_r(n) = \sum_{\zeta \in \mathfrak{R}(r)} \zeta^n = \sum_{d|\gcd(r,n)} d \mu\left(\frac{r}{d}\right)$$

where $n \in \mathbb{Z}$ and $r > 0$.

Fix a \mathbb{Z}_+ -graded vector space V , let

$$B = \{ (k, s) \in \mathbb{Z}_+^2 \mid 1 \leq s \leq \dim V(k) \}$$

and for each $k \geq 0$, choose a basis $\{v_s^k\}_{1 \leq s \leq \dim V(k)}$ for $V(k)$. For any $r > 0$ and $k \geq 0$, let

$$D_{r,k} = \{ ((k_1, s_1), \dots, (k_r, s_r)) \in B^r \mid \sum_{i=1}^r k_i = k \}.$$

The elements of $D_{r,k}$ parameterise a graded basis of $V^r(k)$:

$$D_{r,k} \ni ((k_1, s_1), \dots, (k_r, s_r)) = I \quad \leftrightarrow \quad v_I = v_{s_1}^{k_1} \otimes \dots \otimes v_{s_r}^{k_r} \in V(k).$$

Define an automorphism τ_r of the sets $D_{r,k}$ via the rule

$$\tau_r : ((k_1, s_1), \dots, (k_r, s_r)) \mapsto ((k_r, s_r), (k_1, s_1), \dots, (k_{r-1}, s_{r-1})).$$

The automorphisms σ_r and τ_r are compatible in the sense that

$$\sigma_r(v_I) = v_{\tau_r(I)}, \quad I \in D_{r,k}, \quad k \geq 0.$$

For $I \in D_{r,k}$, write $\text{ord } I = d$ for the minimal positive integer such that $(\tau_r)^d(I) = I$. For any positive divisor d of r , let

$$O_{r,d}(k) = \# \{ I \in D_{r,k} \mid \text{ord } I = d \}, \quad k \geq 0,$$

and write $\mathcal{O}_{r,d}(X) = \sum_{k \geq 0} O_{r,d}(k) X^k$ for the generating function. It is apparent that

$$(5.3) \quad \mathcal{P}_{V^r}(X) = (\mathcal{P}_V(X))^r = \sum_{d \mid r} \mathcal{O}_{r,d}(X)$$

Lemma 5.4. Suppose that l, r are positive integers and that $l \mid r$. Then

$$\{ d \mid d > 0, \quad r/l \mid d \text{ and } d \mid r \} = \{ r/d' \mid d' > 0, \quad d' \mid l \}.$$

Proof. If $d > 0$ and $\frac{r}{l} \mid d$, then there exists some positive integer s such that

$$d = \frac{r}{l} s = \frac{r}{(l/s)};$$

if in addition $d \mid r$, then $d' := l/s$ is a positive integer, and so $d = r/d'$ with $d' \mid l$. Conversely, if $d' \mid l$, then $r/l \mid r/d'$, and it is obvious that $r/d' \mid r$. \square

Proposition 5.5. For any \mathbb{Z}_+ -graded vector space V and any $r > 0$,

$$\mathcal{P}_{V_n^r}(X) = \frac{1}{r} \sum_{d \mid \gcd(r,n)} d \mathcal{O}_{r, \frac{r}{d}}(X).$$

for all $n \in \mathbb{Z}$.

Proof. Suppose that $k \geq 0$, and write $D_{r,k} = \bigsqcup_{O \in \mathcal{P}} O$ for the decomposition of $D_{r,k}$ into a disjoint union of orbits for the action of \mathbb{Z}_r defined by τ_r . Then

$$V^r(k) = \bigoplus_{O \in \mathcal{P}} U_O, \quad U_O = \text{span} \{ v_I \mid I \in O \},$$

and moreover $\sigma_r(U_O) = U_O$. For any orbit $O \in \mathcal{P}$, the action of σ_r on U_O defines the regular representation of \mathbb{Z}_d , where $d = \#O$ is the size of the orbit; in particular, the eigenvalues of σ_r on U_O are precisely the roots of unity ζ such that $\zeta^d = 1$, each with multiplicity 1. Now ζ_r^n is of order $\frac{r}{\gcd(r,n)}$. Therefore,

$$\begin{aligned} \dim V_n^r(k) &= \# \{ O \in \mathcal{P} \mid \frac{r}{\gcd(r,n)} \mid \#O \} \\ &= \# \{ O \in \mathcal{P} \mid \#O = \frac{r}{d} \text{ for some } d \mid \gcd(r,n) \} \end{aligned}$$

where the last equality follows from Lemma 5.4 with $l = \gcd(r,n)$. The number of orbits $O \in \mathcal{P}$ of size r/d is precisely $d/r \cdot O_{r,r/d}(k)$. It follows therefore that

$$\dim V_n^r(k) = \sum_{d \mid \gcd(r,n)} \frac{d}{r} O_{r,\frac{r}{d}}(k),$$

which yields the required equality of generating functions. \square

Proposition 5.6. For any \mathbb{Z}_+ -graded vector space V and positive integers r, d with $d \mid r$,

$$\mathcal{O}_{r,d}(X) = \mathcal{O}_{d,d}(X^{\frac{r}{d}})$$

Proof. Suppose that $k \geq 0$, that

$$I = ((k_1, s_1), \dots, (k_r, s_r)) \in D_{r,k}$$

and that $\text{ord } I = d$. Then

$$I' = ((k_1, s_1), \dots, (k_d, s_d)) \in D_{d, \frac{kd}{r}}.$$

and $\text{ord } I' = d$. This establishes a bijection between order- d elements of the sets $D_{r,k}$ and $D_{d, \frac{kd}{r}}$, and so $O_{r,d}(k) = O_{d,d}(\frac{kd}{r})$. Therefore

$$\begin{aligned} \mathcal{O}_{r,d}(X) &= \sum_{k \geq 0} O_{d,d}(\frac{kd}{r}) X^k \\ &= \sum_{k \geq 0} O_{d,d}(k) (X^{\frac{r}{d}})^k \\ &= \mathcal{O}_{d,d}(X^{\frac{r}{d}}). \end{aligned}$$

\square

It follows immediately from Proposition 5.6 and equation (5.3) that

$$(5.7) \quad (\mathcal{P}_V(X))^r = \sum_{d \mid r} \mathcal{O}_{d,d}(X^{\frac{r}{d}}).$$

Proposition 5.8. For any \mathbb{Z}_+ -graded vector space V and $r > 0$,

$$\mathcal{O}_{r,r}(X) = \sum_{d|r} \mu(d) \left(\mathcal{P}_V(X^d) \right)^{\frac{r}{d}}.$$

Proof. The claim is trivial if $r = 1$, so suppose that $s > 1$ and that the claim holds for all $0 < r < s$. Then:

$$\begin{aligned} \mathcal{O}_{s,s}(X) &= (\mathcal{P}_V(X))^s - \sum_{d|s, d \neq s} \mathcal{O}_{d,d}(X^{\frac{s}{d}}) \quad (\text{by equation (5.7)}) \\ &= (\mathcal{P}_V(X))^s - \sum_{d|s, d \neq s} \sum_{d'|d} \mu(d') \left(\mathcal{P}_V(X^{\frac{sd'}{d}}) \right)^{\frac{d}{d'}} \quad (\text{by inductive hypothesis}) \\ &= (\mathcal{P}_V(X))^s - \sum_{e|s, e \neq 1} \left(\sum_{d|e, d \neq e} \mu(d) \right) (\mathcal{P}_V(X^e))^{\frac{s}{e}} \\ &\quad (\text{write } e = \frac{sd'}{d} \text{ and use Lemma 5.4}) \\ &= (\mathcal{P}_V(X))^s - \sum_{e|s, e \neq 1} (-\mu(e)) (\mathcal{P}_V(X^e))^{\frac{s}{e}} \quad (\text{by equation (5.1)}) \\ &= \sum_{e|s} \mu(e) (\mathcal{P}_V(X^e))^{\frac{s}{e}}, \end{aligned}$$

and so the claim holds for s also. □

Theorem 5.9. For any \mathbb{Z}_+ -graded vector space V , $r > 0$ and $n \in \mathbb{Z}$,

$$\mathcal{P}_{V_n^r}(X) = \frac{1}{r} \sum_{d|r} c_d(n) \left(\mathcal{P}_V(X^d) \right)^{\frac{r}{d}}.$$

Proof. For any $n \in \mathbb{Z}$,

$$\begin{aligned}
\mathcal{P}_{V_n^r}(X) &= \frac{1}{r} \sum_{d|\gcd(r,n)} d \mathcal{O}_{r, \frac{r}{d}}(X) \quad (\text{by Proposition 5.5}) \\
&= \frac{1}{r} \sum_{d|\gcd(r,n)} d \mathcal{O}_{\frac{r}{d}, \frac{r}{d}}(X^d) \quad (\text{by Proposition 5.6}) \\
&= \frac{1}{r} \sum_{d|\gcd(r,n)} d \sum_{d'|\frac{r}{d}} \mu(d') \left(\mathcal{P}_V(X^{dd'}) \right)^{\frac{r}{dd'}} \quad (\text{by Proposition 5.8}) \\
&= \frac{1}{r} \sum_{e|r} \left(\sum_{d|\gcd(e,n)} d \mu\left(\frac{e}{d}\right) \right) (\mathcal{P}_V(X^e))^{\frac{r}{e}} \\
&= \frac{1}{r} \sum_{e|r} c_e(n) (\mathcal{P}_V(X^e))^{\frac{r}{e}},
\end{aligned}$$

where the last equality follows from equation (5.2). \square

6. CHARACTER FORMULAE

In this section, we show that if $\varphi \in \mathcal{E}$, then the module $\mathfrak{L}(\varphi)$ is an irreducible highest-weight module for the truncated current Lie algebra $\mathfrak{g}(\varphi)$. An explicit formula for the character of such a module was obtained in [13]. Therefore, we are able to derive an explicit formula for $\text{char } \mathbf{N}(\varphi)$ by employing the results of Sections 4 and 5.

6.1. Modules for truncated current Lie algebras. Associated to any Lie algebra \mathfrak{a} and non-zero $\varphi \in \mathcal{E}$ is the *truncated current Lie algebra* $\mathfrak{a}(\varphi)$,

$$\mathfrak{a}(\varphi) = \mathfrak{a} \otimes \frac{\mathcal{A}}{c_\varphi \mathcal{A}},$$

with the Lie bracket given by equation (3.1).

Proposition 6.1. Suppose that $\varphi \in \mathcal{E}$ is non-zero. Then the defining ideal $\mathfrak{g} \otimes c_\varphi \mathcal{A} \subset \hat{\mathfrak{g}}$ acts trivially on the $\hat{\mathfrak{g}}$ -module $\mathfrak{L}(\varphi)$, and so $\mathfrak{L}(\varphi)$ is a $\mathfrak{g}(\varphi)$ -module.

Proof. Let $\mathbb{k}v_+$ denote the one-dimensional $\hat{\mathfrak{h}}$ -module defined by

$$h \otimes a \cdot v_+ = (a \cdot \varphi)(0)v_+, \quad a \in \mathcal{A}.$$

Then by definition of the characteristic polynomial c_φ , the subalgebra $\mathfrak{h} \otimes c_\varphi \mathcal{A} \subset \hat{\mathfrak{h}}$ acts trivially upon v_+ , and so $\mathbb{k}v_+$ may be considered as an $\mathfrak{h}(\varphi)$ -module. Let $\mathfrak{g}_+(\varphi) \cdot v_+ = 0$, and let

$$M = \text{Ind}_{\mathfrak{h}(\varphi) + \mathfrak{g}_+(\varphi)}^{\mathfrak{g}(\varphi)} \mathbb{k}v_+$$

denote the induced $\mathfrak{g}(\varphi)$ -module. Denote by L the unique irreducible quotient of M . Then L is a $\hat{\mathfrak{g}}$ -module, via the canonical epimorphism $\hat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}(\varphi)$, and is irreducible with highest-weight defined by the function φ . Hence $\mathfrak{L}(\varphi) \cong L$ as $\hat{\mathfrak{g}}$ -modules, and the claim follows from the construction of L . \square

6.2. Tensor products.

Proposition 6.2. Let $\varphi_1, \varphi_2 \in \mathcal{E}$. Then

$$\mathfrak{L}(\varphi_1 + \varphi_2) \cong \mathfrak{L}(\varphi_1) \otimes \mathfrak{L}(\varphi_2),$$

as $\hat{\mathfrak{g}}$ -modules if c_{φ_1} and c_{φ_2} are co-prime.

Proof. Let $\varphi = \varphi_1 + \varphi_2$. Then $c_\varphi = c_{\varphi_1} c_{\varphi_2}$ since c_{φ_1} and c_{φ_2} are co-prime. By Proposition 6.1, $\mathfrak{L}(\varphi)$ is an irreducible module for $\mathfrak{g}(\varphi)$, and by the Chinese Remainder Theorem,

$$(6.3) \quad \mathfrak{g}(\varphi) \cong \mathfrak{g}(\varphi_1) \oplus \mathfrak{g}(\varphi_2).$$

By Proposition 6.1, $\mathfrak{L}(\varphi_i)$ is a module for $\mathfrak{g}(\varphi_i)$, $i = 1, 2$. The Lie algebra $\mathfrak{g}(\varphi_i)$ is finite-dimensional, and \mathbb{k} is algebraically closed, and so $\mathcal{U}(\mathfrak{g}(\varphi_i))$ is Schurian [11], $i = 1, 2$. Thus $\mathcal{U}(\mathfrak{g}(\varphi_i))$ is tensor-simple [1], and so $\mathfrak{L}(\varphi_1) \otimes \mathfrak{L}(\varphi_2)$ is an irreducible module for $\mathcal{U}(\mathfrak{g}(\varphi_1)) \otimes \mathcal{U}(\mathfrak{g}(\varphi_2))$. The decomposition (6.3) and the Poincaré-Birkhoff-Witt Theorem imply that

$$\mathcal{U}(\mathfrak{g}(\varphi_1)) \otimes \mathcal{U}(\mathfrak{g}(\varphi_2)) \cong \mathcal{U}(\mathfrak{g}(\varphi)),$$

and so $\mathfrak{L}(\varphi_1) \otimes \mathfrak{L}(\varphi_2)$ is an irreducible module for $\mathfrak{g}(\varphi)$. The irreducible highest-weight modules $\mathfrak{L}(\varphi)$ and $\mathfrak{L}(\varphi_1) \otimes \mathfrak{L}(\varphi_2)$ are of equal highest weight, by the Leibniz rule, and hence are isomorphic. \square

6.3. Semi-invariants of the modules $\mathfrak{L}(\varphi)$. For any $\varphi \in \mathcal{E}$, consider $\mathfrak{L}(\varphi)$ as a \mathbb{Z}_+ -graded vector space via

$$\mathfrak{L}(\varphi) = \bigoplus_{k \geq 0} \mathfrak{L}(\varphi)(k), \quad \mathfrak{L}(\varphi)(k) = \mathfrak{L}(\varphi)_{\varphi(0)\frac{\mathfrak{g}}{2} - k\alpha}, \quad k \geq 0.$$

Proposition 6.4. Suppose that $\varphi \in \mathcal{E}$ is non-zero and that $\varphi = \wp_r \psi$, where $r = \deg \varphi$ and $\psi \in \mathcal{E}$, as per Lemma 2.10. Then there exists an isomorphism

$$\Omega : \mathfrak{L}(\varphi) \rightarrow \mathfrak{L}(\psi)^r$$

of \mathbb{Z}_+ -graded vector spaces such that $\sigma_r = \Omega \circ \eta_\varphi \circ \Omega^{-1}$.

Proof. For $j \in \mathbb{Z}_r$, write $\psi^j = \text{EXP}(\zeta_r^{-j})\psi$. Then $c_\varphi = \prod_{j \in \mathbb{Z}_r} c_{\psi^j}$ is a decomposition of c_φ into co-prime factors, and $\varphi = \sum_{j \in \mathbb{Z}_r} \psi^j$. By the Chinese Remainder Theorem, there exists a finite linearly independent set $\{a_i \mid i \in I\} \subset \mathcal{A}$ such that $\{a_i + c_\psi \mathcal{A} \mid i \in I\}$ is a basis for $\mathcal{A}/c_\psi \mathcal{A}$ and

$$a_i \equiv 0 \pmod{c_{\psi^j}}, \quad j \not\equiv 0 \pmod{r}, \quad i \in I, \quad j \in \mathbb{Z}_r.$$

Write $a_{i,j}(t) = a_i(\zeta_r^{-j}t)$, $i \in I$, $j \in \mathbb{Z}_r$. Then by symmetry, $\{a_{i,j} + c_{\psi^j} \mathcal{A} \mid i \in I\}$ is a basis for $\mathcal{A}/c_{\psi^j} \mathcal{A}$ and

$$a_{i,j} \equiv 0 \pmod{c_{\psi^k}}, \quad j \not\equiv k \pmod{r}, \quad i \in I, \quad j, k \in \mathbb{Z}_r.$$

For any $i \in I$ and $j \in \mathbb{Z}_r$,

$$(6.5) \quad \eta_\varphi(f \otimes a_{i,j} \cdot w) = f \otimes a_{i,j}(\zeta_r^{-1}t) \cdot \eta_\varphi(w) = f \otimes a_{i,j+1} \cdot \eta_\varphi(w), \quad w \in \mathfrak{L}(\varphi).$$

By Proposition 6.2, there exists an isomorphism

$$\Upsilon : \mathfrak{L}(\varphi) \rightarrow \bigotimes_{j \in \mathbb{Z}_r} \mathfrak{L}(\psi^j)$$

of $\hat{\mathfrak{g}}$ -modules, and we may assume that $\Upsilon(v_\varphi) = \bigotimes_{j \in \mathbb{Z}_r} v_{\psi^j}$. For any $k \in \mathbb{Z}_r$, identify

$$(6.6) \quad \mathfrak{L}(\psi^k) = 1 \otimes \cdots \otimes \mathfrak{L}(\psi^k) \otimes \cdots \otimes 1 \subset \bigotimes_{j \in \mathbb{Z}_r} \mathfrak{L}(\psi^j).$$

Then $\mathfrak{L}(\psi^k)$ is generated by the action of the basis $\{f \otimes a_{i,k} \mid i \in I\}$ of $\mathfrak{g}_-(\psi^k)$ on the highest-weight vector $\Upsilon(v_\varphi)$. Therefore, modulo the identification (6.6),

$$(\Upsilon \circ \eta_\varphi \circ \Upsilon^{-1})(\mathfrak{L}(\psi^k)) \subset \mathfrak{L}(\psi^{k+1}), \quad k \in \mathbb{Z}_r,$$

by equation (6.5). Since η_φ is an automorphism of the \mathbb{Z}_+ -graded vector space $\mathfrak{L}(\varphi)$, the restriction

$$(\Upsilon \circ \eta_\varphi \circ \Upsilon^{-1}) : \mathfrak{L}(\psi^k) \rightarrow \mathfrak{L}(\psi^{k+1}),$$

is an isomorphism of the \mathbb{Z}_+ -graded vector spaces. These isomorphisms obviously induce isomorphisms $\epsilon_j : \mathfrak{L}(\psi^j) \rightarrow \mathfrak{L}(\psi^0) = \mathfrak{L}(\psi)$, and

$$\epsilon_j : \prod_{i \in I} (f \otimes a_{i,j})^{k_i} \cdot v_{\psi^j} \mapsto \prod_{i \in I} (f \otimes a_i)^{k_i} \cdot v_\psi,$$

by equation (6.5). Let $\epsilon = \bigotimes_{j \in \mathbb{Z}_r} \epsilon_j$, and write Ω for the composition

$$\epsilon \circ \Upsilon : \mathfrak{L}(\varphi) \rightarrow \mathfrak{L}(\psi)^r.$$

The vector space $\mathfrak{L}(\psi)^r$ is spanned by the homogeneous tensors

$$\bigotimes_{j \in \mathbb{Z}_r} \prod_{i \in I} (f \otimes a_i)^{k_{i,j}} v_{\psi}, \quad k_{i,j} \geq 0.$$

For any homogeneous tensor of this form

$$\begin{aligned} (\Omega \circ \eta_{\varphi} \circ \Omega^{-1}) \cdot \left(\bigotimes_{j \in \mathbb{Z}_r} \prod_{i \in I} (f \otimes a_i)^{k_{i,j}} v_{\psi} \right) &= \epsilon \circ (\Upsilon \circ \eta_{\varphi} \circ \Upsilon^{-1}) \left(\bigotimes_{j \in \mathbb{Z}_r} \prod_{i \in I} (f \otimes a_{i,j})^{k_{i,j}} v_{\psi^j} \right) \\ &= \epsilon \circ (\Upsilon \circ \eta_{\varphi} \circ \Upsilon^{-1}) \left(\prod_{j \in \mathbb{Z}_r} \prod_{i \in I} (f \otimes a_{i,j})^{k_{i,j}} \cdot \bigotimes_{j \in \mathbb{Z}_r} v_{\psi^j} \right) \\ &= \epsilon \left(\prod_{j \in \mathbb{Z}_r} \prod_{i \in I} (f \otimes a_{i,j})^{k_{i,j-1}} \cdot \bigotimes_{j \in \mathbb{Z}_r} v_{\psi^j} \right) \\ &= \epsilon \left(\bigotimes_{j \in \mathbb{Z}_r} \prod_{i \in I} (f \otimes a_{i,j})^{k_{i,j-1}} v_{\psi^j} \right) \\ &= \bigotimes_{j \in \mathbb{Z}_r} \prod_{i \in I} (f \otimes a_i)^{k_{i,j-1}} v_{\psi} \\ &= \sigma_r \left(\bigotimes_{j \in \mathbb{Z}_r} \prod_{i \in I} (f \otimes a_i)^{k_{i,j}} v_{\psi} \right), \end{aligned}$$

where the second and fourth equalities are by construction of the polynomials $a_{i,j}$ and the Leibniz rule. Therefore $\Omega \circ \eta_{\varphi} \circ \Omega^{-1} = \sigma_r$ as required. \square

6.4. Character Formulae.

Theorem 6.7. Suppose that $a \in \mathcal{F}$ is a polynomial function and that $\varphi = a \text{EXP}(\lambda)$ for some $\lambda \in \mathbb{k}^{\times}$. Then

$$\mathcal{P}_{\mathfrak{L}(\varphi)}(X) = \begin{cases} \frac{1-X^{a+1}}{1-X} & \text{if } a \in \mathbb{Z}_+, \\ \frac{1}{1-X} & \text{otherwise.} \end{cases}$$

Proof. Let $N = \deg a$, and write $\varphi = \sum_{k=0}^N a_k \theta_{\lambda,k}$. By Proposition 6.1, $\mathfrak{L}(\varphi)$ is a module for the truncated current Lie algebra $\mathfrak{g}(\varphi)$. The Cartan subalgebra of $\mathfrak{g}(\varphi)$ has a basis

$$\{ h \otimes (t - \lambda)^k \mid 0 \leq k \leq N \}.$$

By Lemma 2.1, $h \otimes (t - \lambda)^N$ acts on the highest-weight vector v_{φ} by the scalar

$$(6.8) \quad ((t - \lambda)^N \cdot \varphi)(0) = N! \lambda^N a_N.$$

If $N = 0$, then (6.8) takes the value $a \in \mathbb{k}$, and so $\mathfrak{L}(\varphi)$ is the irreducible \mathfrak{g} -module of highest weight a . Therefore

$$\mathcal{P}_{\mathfrak{L}(\varphi)}(X) = \begin{cases} \frac{1-X^{a+1}}{1-X} & \text{if } a \in \mathbb{Z}_+, \\ \frac{1}{1-X} & \text{otherwise.} \end{cases}$$

If $N > 0$, then a_N is non-zero; thus (6.8) is non-zero and the claim follows from Proposition A.1 of [13]. \square

Suppose that $\varphi \in \mathcal{E}$ is non-zero, $\deg \varphi = r$, and that $\psi \in \mathcal{E}$ is given by Lemma 2.10. Then

$$(6.9) \quad \psi = \sum_i a_i \text{EXP}(\lambda_i)$$

for some finite collection of polynomial functions $a_i \in \mathcal{F}$ and distinct $\lambda_i \in \mathbb{k}^\times$, such that if $(\lambda_i/\lambda_j)^r = 1$, then $i = j$.

Theorem 6.10. Suppose that $\varphi \in \mathcal{E}$ is non-zero, $\deg \varphi = r$, and that

$$\varphi = \wp_r \sum_i a_i \text{EXP}(\lambda_i)$$

where the $a_i \in \mathcal{F}$ and $\lambda_i \in \mathbb{k}^\times$ are given by (6.9). Let

$$P_\varphi(Z) = \frac{\prod_{a_i \in \mathbb{Z}_+} (1 - Z^{a_i+1})}{(1 - Z)^M},$$

where $M = \sum_i (\deg a_i + 1)$ and the product is over those indices i such that $a_i \in \mathbb{Z}_+$. Then

$$\text{char } \mathbf{N}(\varphi) = Z^{\varphi(0)\frac{\alpha}{2}} \cdot \frac{1}{r} \sum_{n \in \mathbb{Z}} \sum_{d|r} c_d(n) \left(P_\varphi(Z^{-d\alpha}) \right)^{\frac{r}{d}} Z^{n\delta}.$$

Proof. By Corollary 4.5 and Proposition 6.4,

$$\text{char } \mathbf{N}(\varphi) = Z^{\varphi(0)\frac{\alpha}{2}} \cdot \sum_{n \in \mathbb{Z}} \mathcal{P}_{\mathfrak{L}(\psi)_n^r}(Z^{-\alpha}) Z^{n\delta},$$

and by Theorem 5.9,

$$(6.11) \quad \mathcal{P}_{\mathfrak{L}(\psi)_n^r}(X) = \frac{1}{r} \sum_{d|r} c_d(n) \left(\mathcal{P}_{\mathfrak{L}(\psi)}(X^d) \right)^{\frac{r}{d}}.$$

By Proposition 6.2, there is an isomorphism of $\hat{\mathfrak{g}}$ -modules

$$\mathfrak{L}(\psi) \cong \bigotimes_i \mathfrak{L}(\psi^i), \quad \psi^i = a_i \text{EXP}(\lambda_i),$$

since the λ_i are distinct. In particular, $\mathcal{P}_{\mathfrak{L}(\psi)} = \prod_i \mathcal{P}_{\mathfrak{L}(\psi^i)}$, and so

$$\mathcal{P}_{\mathfrak{L}(\psi)}(X) = P_{\varphi}(X)$$

by Theorem 6.7. Therefore the claim follows from equation (6.11). \square

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